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NEW INVESTIGATIONS OF THE OPTIMUM RENDEZVOUS MANEUVER OF SPACE VEHICLES ON ELLIPTICAL TRAJECTORIES FOR MINIMUM FUEL CONSUMPTION

Alexe Marinescu, Stefan Staicu and Vladimir Cardos

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16. Abstract The optimal rendezvous manoeuvre of two space vehicles on elliptic flight paths with minimum fuel consumption is investigated. The target body shall move on an elliptic flight path as a satellite of the earth or another planet and the following (propelled) body on a neighbouring path. The variational problem for the manoeuvre is formulated as an extremum task with constraints with considering the linear and the nonlinear equations of motion for the propelled flying body. Exact and approximative expressions for the optimal variational laws of the flight parameters of the propelled flying body are derived due to which numerical applications are performed employing electronic computers.

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NEW INVESTIGATIONS OF THE OPTIMUM RENDEZVOUS MANEUVER OF SPACE VEHICLES ON ELLIPTICAL TRAJECTORIES FOR MINIMUM FUEL CONSUMPTION

Alexe Marinescu*, Stefan Staicu **, Vladimir Cardos***

1. Introduction

It is difficult to obtain analytical solutions to the optimum rendezvous maneuver problem for minimum fuel consumption and assuming elliptical trajectories, if variational calculus is used. ****/146

In spite of the various formulations of the variational problem for the rendezvous maneuver on such trajectories, only a few authors were able to finding analytical solutions for the optimum variation laws of the flight parameters during the powered phase. As far as we know, no numerical results of any practical interest have been published. In the following we will treat the rendezvous maneuver for minimum fuel consumption on elliptical trajectories. This is based on the general formulation of the variational problem for the rendezvous maneuver on arbitrary conic section trajectories which was developed by Al Marinescu in [1]. We will give a compact but very complete description of the problem. We will emphasize the most advantageous solution paths for optimum motion of the propelled spacecraft in a critical investigation. We will then obtain useful numerical results. The hypotheses and approximations

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which are used as a basis for the differential equations for determining the extremal functions were specified in [1,2]. Also, we should point out that the accuracy improves if the eccentricity of the trajectory is small, i. e. the thrust phase can be longer the smaller the eccentricity.

The results can be used for large eccentricities ($\varepsilon \geq 0.1$) during the contact phase because of the approximations. For small eccentricities ($\varepsilon < 0.1$) it can be used for the contact phase as well as for the approach phase.

2. Notation

OXZY	Inertial system with origin at the planet center (Fig. 1)
Axzy	Target fixed (orbital) coordinate system
x, z, y	Coordinates of the propelled spacecraft in the orbital system
r_o (t)	Radius vector of the target body
r_{op}	Radius vector to perigee
p	Focal parameter of the trajectory of the target body
ε	Eccentricity of the target body trajectory
μ	Gravity parameter
ϕ (t)	True anomaly of the target body
t	Time
τ	Burn duration
V_x, V_z, V_y	Components of the relative velocity of the propelled spacecraft in the orbital system
a_x, a_z, a_y	Components of the thrust acceleration in the orbital system
g_x, g_z, g_y	Components of the gravity acceleration of the orbital system
a_{xr}, a_{zr}, a_{yr}	Components of relative (total) acceleration in the orbital system
r_{01}, φ_1	Coordinates of the target body at the beginning of the maneuver (Fig. 2)

Notation - cont'd

r_{02}, φ_2	Coordinates of the target body at the end of the maneuver (at the contact time)
r_{0m}, φ_m	Average coordinates of the target body $[= (r_{01} + r_{02})/2, = (\varphi_1 + \varphi_2)/2]$
$\lambda_1, \ldots, \lambda_6$	Lagrange Multipliers.

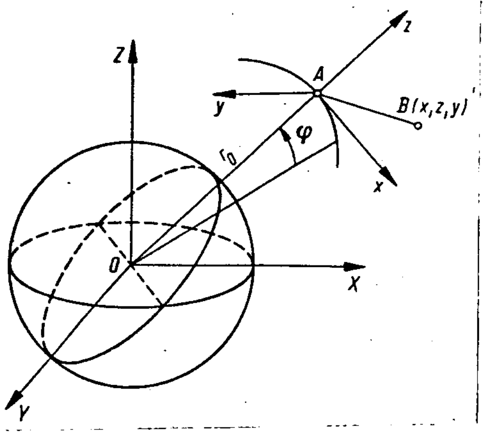

Fig. 1

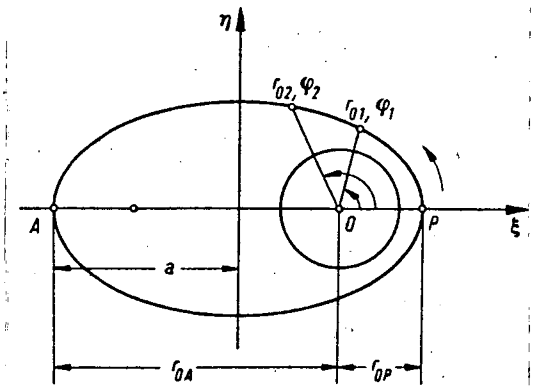

Fig. 2

3. Investigations of the optimum Rendezvous Maneuver based on linear Equations of Motion of the Powered Spacecraft (linear theory)

Just as was shown in [1] within the framework of the linear theory, the variational problem of optimum rendezvous maneuver for minimum fuel consumption on the elliptical trajectories ($0<\varepsilon<1$) amounts to finding the minimum of the functional

$$J = \int_0^\tau (a_x^2 + a_z^2 + a_y^2)\, dt \tag{1}$$

with the conditions

$$\left\{\begin{aligned} \Phi_1 &= \frac{dx}{dt} - V_x = 0, \\ \Phi_2 &= \frac{dV_x}{dt} - \alpha_1 x + \alpha_2 z - \alpha_3 V_z - a_x = 0, \\ \Phi_3 &= \frac{dz}{dt} - V_z = 0, \end{aligned}\right. \tag{2}$$

$$\left\{\begin{aligned} \Phi_4 &= \frac{dV_z}{dt} - \alpha_2 x - \beta_1 z + \alpha_3 V_x - a_z = 0, \\ \Phi_5 &= \frac{dy}{dt} - V_y = 0, \\ \Phi_6 &= \frac{dV_y}{dt} + \gamma_1 y - a_y = 0 \end{aligned}\right. \tag{2 cont.}$$

where

$$\left\{\begin{aligned} \alpha_1 &= \frac{\mu p}{r_{0m}^4} - \frac{\mu}{r_{0m}^3}, \\ \alpha_2 &= \frac{2\mu\varepsilon}{r_{0m}^3} \sin\varphi_m, \\ \alpha_3 &= 2\frac{\sqrt{\mu p}}{r_{0m}^2}, \\ \beta_1 &= \frac{\mu p}{r_{0m}^4} + \frac{2\mu}{r_{0m}^3}, \\ \gamma_1 &= \frac{\mu}{r_{0m}^3} \end{aligned}\right. \tag{3}$$

Using the method described in [1,2], we obtain the following differential equations for the extreme curve:

$$\left\{\begin{aligned} &\frac{d\lambda_1}{dt} = -\alpha_1\lambda_2 - \alpha_2\lambda_4, && \frac{d\lambda_2}{dt} = -\lambda_1 + \alpha_3\lambda_4, \\ &\frac{d\lambda_3}{dt} = \alpha_2\lambda_2 - \beta_1\lambda_4, && \frac{d\lambda_4}{dt} = -\alpha_3\lambda_2 - \lambda_3, \\ &\frac{d\lambda_5}{dt} = \gamma_1\lambda_6, && \frac{d\lambda_6}{dt} = -\lambda_5, \\ &\frac{dx}{dt} = V_x, \quad \frac{dV_x}{dt} = \alpha_1 x - \alpha_2 z + \alpha_3 V_z + a_x, \\ &\frac{dz}{dt} = V_z, \quad \frac{dV_z}{dt} = \alpha_2 x + \beta_1 z - \alpha_3 V_x + a_z, \\ &\frac{dy}{dt} = V_y, \quad \frac{dV_y}{dt} = -\gamma_1 y + a_y \end{aligned}\right. \tag{4}$$

We also obtain the following algebraic equations

$$2a_x - \lambda_2 = 0, \quad 2a_z - \lambda_4 = 0, \quad 2a_y - \lambda_6 = 0. \tag{5}$$

The solutions of the systems (4) and (5) represent the optimum variational laws of the motion parameters of the powered spacecraft, which characterize the optimum rendezvous maneuver on elliptical trajectories for minimum fuel consumption. /148

3.1 Exact Solutions

We will now attempt exact integration of the differential equation system (4). First of all we should note that the fifth and sixth equation lead to

$$\frac{d^2\lambda_6}{dt^2} + \gamma_1 \lambda_6 = 0 \tag{6}$$

and the eleventh and twelfth equation lead to

$$\frac{d^2 y}{dt^2} + \gamma_1 y = a_y \tag{7}$$

These equations can be solved without any difficulty.

The other equations can be grouped into two different systems. Classical methods are used for their integration.

The following system is the result of the first four equations

$$\left\{\begin{array}{l} -\dfrac{d\lambda_1}{dt} - \alpha_1\lambda_2 - \alpha_2\lambda_4 = 0, \\ -\dfrac{d\lambda_2}{dt} - \lambda_1 + \alpha_3\lambda_4 = 0 \\ -\dfrac{d\lambda_3}{dt} + \alpha_2\lambda_2 - \beta_1\lambda_4 = 0, \\ -\dfrac{d\lambda_4}{dt} - \alpha_3\lambda_2 - \lambda_3 = 0 \end{array}\right. \tag{8}$$

using the usual substitutions

$$\lambda_1 = Ae^{rt}, \quad \lambda_2 = Be^{rt}, \quad \lambda_3 = Ce^{rt}, \quad \lambda_4 = De^{rt}$$

we find the following characteristic equation

$$r^4 + (\alpha_3^2 - \beta_1 - \alpha_1)\, r^2 + 2\,\alpha_2\,\alpha_3\, r + \alpha_1\,\beta_1 + \alpha_2^2 = 0\,, \tag{9}$$

which has two real roots r_1, r_2 within the range of values of interest in practice. It also has two imaginary roots $r_3 = \nu + i\eta,\ r_4 = \nu - i\eta$. The solutions of system (8) obtained in this way are

$$\left\{\begin{aligned}
\lambda_1 &= m_{11}\, C_1\, e^{r_1 t} + m_{12}\, C_2\, e^{r_2 t} + \\
&\quad + (a_1 \cos\eta\, t - b_1 \sin\eta\, t)\, C_3\, e^{\nu t} + \\
&\qquad + (a_1 \sin\eta\, t + b_1 \cos\eta\, t)\, C_4\, e^{\nu t}\,, \\
\lambda_2 &= m_{21}\, C_1\, e^{r_1 t} + m_{22}\, C_2\, e^{r_2 t} + \\
&\quad + (a_2 \cos\eta\, t - b_2 \sin\eta\, t)\, C_3\, e^{\nu t} + \\
&\qquad + (a_2 \sin\eta\, t + b_2 \cos\eta\, t)\, C_4\, e^{\nu t}\,, \\
\lambda_3 &= m_{31}\, C_1\, e^{r_1 t} + m_{32}\, C_2\, e^{r_2 t} + \\
&\quad + (a_3 \cos\eta\, t - b_3 \sin\eta\, t)\, C_3\, e^{\nu t} + \\
&\qquad + (a_3 \sin\eta\, t + b_3 \cos\eta\, t)\, C_4\, e^{\nu t}\,, \\
\lambda_4 &= m_{41}\, C_1\, e^{r_1 t} + m_{42}\, C_2\, e^{r_2 t} + \\
&\quad + (a_4 \cos\eta\, t - b_4 \sin\eta\, t)\, C_3\, e^{\nu t} + \\
&\qquad + (a_4 \sin\eta\, t + b_4 \cos\eta\, t)\, C_4\, e^{\nu t}
\end{aligned}\right. \tag{10}$$

with

$$\left\{\begin{aligned}
m_{11} &= -r_1^3 + (\beta_1 - \alpha_3^2)\, r_1 - \alpha_2\,\alpha_3\,, \\
m_{21} &= r_1^2 - \beta_1\,, \\
m_{31} &= \alpha_2\, r_1 + \beta_1\,\alpha_3\,, \quad m_{41} = -\alpha_3\, r_1 - \alpha_2\,, \\
m_{12} &= -r_2^3 + (\beta_1 - \alpha_3^2)\, r_2 - \alpha_2\,\alpha_3\,, \\
m_{22} &= r_2^2 - \beta_1\,, \\
m_{32} &= \alpha_2\, r_2 + \beta_1\,\alpha_3\,, \quad m_{42} = -\alpha_3\, r_2 - \alpha_2\,, \\
a_1 &= -\nu^3 + 3\,\nu\,\eta^2 + (\beta_1 - \alpha_3^2)\,\nu - \alpha_2\,\alpha_3\,, \\
b_1 &= -3\,\nu^2\,\eta + \eta^3 + (\beta_1 - \alpha_3^2)\,\eta\,, \\
a_2 &= \nu^2 - \eta^2 - \beta_1\,, \qquad b_2 = 2\,\nu\,\eta\,, \\
a_3 &= \alpha_2\,\nu + \beta_1\,\alpha_3\,, \qquad b_3 = \alpha_2\,\eta\,, \\
a_4 &= -\nu\,\alpha_3 - \alpha_2\,, \qquad b_4 = -\alpha_3\,\eta\,.
\end{aligned}\right.$$

for the homogeneous system

$$\left\{\begin{aligned}
&\frac{dx}{dt} - V_x = 0\,, \\
&\frac{dV_x}{dt} - \alpha_1\, x + \alpha_2\, z - \alpha_3\, V_z = 0\,, \\
&\frac{dz}{dt} - V_z = 0\,, \\
&\frac{dV_z}{dt} - \alpha_2\, x + \alpha_3\, V_x - \beta_1\, z = 0
\end{aligned}\right. \tag{12}$$

and using the notation

$$x = \bar{A}\, e^{\bar{r} t}, \quad V_x = \bar{B}\, e^{\bar{r} t}, \quad z = \bar{C}\, e^{\bar{r} t}, \quad V_z = \bar{D}\, e^{\bar{r} t}$$

we obtain the following characteristic equation

$$(13)\quad \bar{r}^4 + (\alpha_3^2 - \beta_1 - \alpha_1)\, \bar{r}^2 - 2\,\alpha_2\,\alpha_3\,\bar{r} + \alpha_1\,\beta_1 + \alpha_2^2 = 0\,. \tag{13}$$

Since the characteristic equation (13) is only different from the characteristic equation (9) because of the sign of the coefficient of $\bar{r}$, we obtain the following result from the theory of algebraic equations

$$\begin{cases} \bar{r}_1 = -r_1\,, & \bar{r}_2 = -r_2\,, \\ \bar{r}_3 = -\nu - i\,\eta\,, & \bar{r}_4 = -\nu + i\,\eta\,. \end{cases} \tag{14}$$

In this case the solutions of system (12) are given by

$$\begin{cases} x = n_{11}\, C_5\, e^{-r_1 t} + n_{12}\, C_6\, e^{-r_2 t} + \\ \quad + (c_1 \cos\eta\, t + d_1 \sin\eta\, t)\, C_7\, e^{-\nu t} + \\ \qquad + (-\, c_1 \sin\eta\, t + d_1 \cos\eta\, t)\, C_8\, e^{-\nu t}\,, \\ V_x = n_{21}\, C_5\, e^{-r_1 t} + n_{22}\, C_6\, e^{-r_2 t} + \\ \quad + (c_2 \cos\eta\, t + d_2 \sin\eta\, t)\, C_7\, e^{-\nu t} + \\ \qquad + (-\, c_2 \sin\eta\, t + d_2 \cos\eta\, t)\, C_8\, e^{-\nu t}\,, \\ z = n_{31}\, C_5\, e^{-r_1 t} + n_{32}\, C_6\, e^{-r_2 t} + \\ \quad + (c_3 \cos\eta\, t + d_3 \sin\eta\, t)\, C_7\, e^{-\nu t} + \\ \qquad + (-\, c_3 \sin\eta\, t + d_3 \cos\eta\, t)\, C_8\, e^{-\nu t}\,, \\ V_z = n_{41}\, C_5\, e^{-r_1 t} + n_{42}\, C_6\, e^{-r_2 t} + \\ \quad + (c_4 \cos\eta\, t + d_4 \sin\eta\, t)\, C_7\, e^{-\nu t} + \\ \qquad + (-\, c_4 \sin\eta\, t + d_4 \cos\eta\, t)\, C_8\, e^{-\nu t} \end{cases} \tag{15}$$

with

$$
\left\{\begin{aligned}
&n_{11} = -r_1^3 - (\alpha_3^2 - \beta_1) r_1 - \alpha_2 \alpha_3 ,\\
&n_{21} = \alpha_1 r_1^2 - \alpha_2 \alpha_3 r_1 - (\alpha_1 \beta_1 + \alpha_2^2) ,\\
&n_{31} = -r_1 \alpha_2 - \alpha_1 \alpha_2 , \; n_{41} = r_1^2 \alpha_2 + \alpha_1 \alpha_3 r_1 ,\\
&n_{12} = -r_2^3 - (\alpha_3^2 - \beta_1) r_2 - \alpha_2 \alpha_3 ,\\
&n_{22} = \alpha_1 r_1^2 - \alpha_2 \alpha_3 r_2 - (\alpha_1 \beta_1 + \alpha_2^2) ,\\
&n_{32} = -r_2 \alpha_2 - \alpha_1 \alpha_3 , \; n_{42} = r_2^2 \alpha_2 + \alpha_1 \alpha_3 r_2 ,\\
&c_1 = a_1 , \; d_1 = b_1 ,\\
&c_2 = \alpha_1 \nu^2 - \alpha_1 \eta^2 - \alpha_2 \alpha_3 \nu - \alpha_1 \beta_1 - \alpha_2^2 ,\\
&d_2 = 2 \nu \eta \alpha_1 - \alpha_2 \alpha_3 \eta ,\\
&c_3 = -\nu \alpha_2 - \alpha_1 \alpha_3 , \; d_3 = -\eta \alpha_2 ,\\
&c_4 = \alpha_2 \nu^2 - \alpha_2 \eta^2 + \alpha_1 \alpha_3 \nu ,\\
&d_4 = 2 \nu \eta \alpha_2 + \alpha_1 \alpha_3 \eta .
\end{aligned}\right. \tag{16}
$$

Considering the solution of equation (6)

$$
\lambda_6 = C_9 \cos \omega t + C_{10} \sin \omega t \qquad (\omega = \sqrt{\gamma_1}) , \tag{17}
$$

of solutions (10) and the algebraic equation (5), we find the following components of thrust acceleration

$$
\left\{\begin{aligned}
a_x = {} & \frac{1}{2} [m_{21} C_1 e^{r_1 t} + m_{22} C_2 e^{r_2 t} + \\
& + (a_2 \cos \eta t - b_2 \sin \eta t) C_3 e^{\nu t} + \\
& + (a_2 \sin \eta t + b_2 \cos \eta t) C_4 e^{\nu t}] ,\\
a_z = {} & \frac{1}{2} [m_{41} C_1 e^{r_1 t} + m_{42} C_2 e^{r_2 t} + \\
& + (a_4 \cos \eta t - b_4 \sin \eta t) C_3 e^{\nu t} + \\
& + (a_4 \sin \eta t + b_4 \cos \eta t) C_4 e^{\nu t}] ,\\
a_y = {} & \frac{1}{2} C_9 \cos \omega t + \frac{1}{2} C_{10} \sin \omega t .
\end{aligned}\right. \tag{18}
$$

These expressions can now be considered as the right terms for the second and fourth equation (12) as well as for the homogeneous equation corresponding to (7). The solutions of the nonhomogeneous equations are derived from the solutions of the homogeneous equations using the variation of constants method. The calculations performed by the authors are quite complicated and it is not easy to follow the solutions.

Considerable computational effort is required to determine the integration constants. The work is even increased because of the solution of the characteristic equations for any change of an individual parameter.

The increased accuracy determined in this way does not mean this method should be used. This is because it is easy to make mistakes due to the great complexity of the calculations. This is why the authors believe that the approximation method described in [2] is more desirable.

3.2 Approximate Solutions

According to the method mentioned, new solutions for systems (4) and (5) are calculated, which were used for numerical applications. An electronic data-processing computer was used.

Without going into detail, which can be found in [2], we will only give one of these new expressions for the optimum variational law of the equations of motion of the propelled spacecraft, which was determined according to this method:

$$
\left.
\begin{aligned}
a_x &= \frac{1}{2}\left(\frac{\alpha_3 C_1}{K} - \frac{\alpha_2 C_2}{K^2}\right) \sin K t - \\
&\quad - \frac{1}{2}\left(\frac{\alpha_3 C_2}{K} + \frac{\alpha_2 C_1}{K^2}\right) \cos K t - \frac{C_3}{2} t + \frac{C_4}{2}, \\
a_z &= \frac{C_1}{2} \cos K t + \frac{C_2}{2} \sin K t, \\
a_y &= \frac{C_5}{2} \cos \omega t + \frac{C_6}{2} \sin \omega t, \\
V_x &= \left(\frac{\alpha_2 M}{2 K^3} - \frac{\alpha_3 N}{2 K^2}\right) t \sin K t - \\
&\quad - \left(\frac{\alpha_2 N}{2 K^3} + \frac{\alpha_3 M}{2 K^2}\right) t \cos K t +
\end{aligned}
\right\} \quad (19)
$$

$$+\left[\alpha_2\left(\frac{N}{K^4}-\frac{C_1}{2K^3}+\frac{C_{10}}{K^2}\right)+\right.$$
$$\left.+\alpha_3\left(\frac{M}{2K^3}-\frac{C_2}{2K^2}+\frac{C_9}{K}\right)\right]\sin Kt+$$
$$+\left[\alpha_2\left(\frac{M}{K^4}+\frac{C_2}{2K^3}+\frac{C_9}{K^2}\right)-\right.$$
$$\left.-\alpha_3\left(\frac{N}{2K^3}+\frac{C_1}{2K^2}+\frac{C_{10}}{K}\right)\right]\cos Kt-$$
$$-\frac{\alpha_2\alpha_3 C_3}{12K^2}t^3+$$
$$+\left(\frac{\alpha_2\alpha_3 C_4}{4K^2}+\frac{\alpha_3^2 C_3}{4K^2}-\frac{C_3}{4}\right)t^2-$$
$$-\left(\alpha_2 C_{11}+\frac{\alpha_3^2 C_4}{2K^2}-\frac{C_4}{2}\right)t+C_{12},$$

$$V_z=\frac{M}{2K}t\sin Kt-\frac{N}{2K}t\cos Kt+$$
$$+C_9\cos Kt+C_{10}\sin Kt+$$
$$+\frac{\alpha_3 C_3}{2K^2}t-\frac{\alpha_3 C_4}{2K^2},$$

$$V_y=\frac{C_5}{4}t\cos\omega t+\frac{C_6}{4}t\sin\omega t+ \qquad (19)$$
$$+\left(\frac{C_5}{4\omega}-\omega C_7\right)\sin\omega t+$$
$$+\left(\omega C_8-\frac{C_6}{4\omega}\right)\cos\omega t,$$

$$x=-\left(\frac{\alpha_2 N}{2K^4}+\frac{\alpha_3 M}{2K^3}\right)t\sin Kt-$$
$$-\left(\frac{\alpha_2 M}{2K^4}-\frac{\alpha_3 N}{2K^3}\right)t\cos Kt+$$
$$+\left[\alpha_2\left(\frac{3M}{2K^5}+\frac{C_2}{2K^4}+\frac{C_9}{K^3}\right)-\right.$$
$$\left.-\alpha_3\left(\frac{N}{K^4}+\frac{C_1}{2K^3}+\frac{C_{10}}{K^2}\right)\right]\sin Kt-$$
$$-\left[\alpha_2\left(\frac{3N}{2K^5}-\frac{C_1}{2K^4}+\frac{C_{10}}{K^3}\right)+\right.$$
$$\left.+\alpha_3\left(\frac{M}{K^4}-\frac{C_2}{2K^3}+\frac{C_9}{K^2}\right)\right]\cos Kt-$$
$$-\frac{\alpha_2\alpha_3 C_3}{48K^2}t^4+$$

$$
\left.\begin{aligned}
&+\left(\frac{\alpha_2\alpha_3 C_4}{12K^2}+\frac{\alpha_3^2 C_3}{12K^2}-\frac{C_3}{12}\right)t^3-\\
&-\left(\frac{\alpha_2 C_{11}}{2}+\frac{\alpha_3^2 C_4}{4K^2}-\frac{C_4}{4}\right)t^2+C_{12}t+C_{13},\\
&z=-\frac{Nt}{2K^2}\sin Kt-\frac{Mt}{2K^2}\cos Kt+\\
&+\left(\frac{M}{2K^3}+\frac{C_9}{K}\right)\sin Kt-\\
&-\left(\frac{N}{2K^3}+\frac{C_{10}}{K}\right)\cos Kt+\\
&\qquad+\frac{\alpha_3 C_3}{4K^2}t^2-\frac{\alpha_3 C_4}{2K^2}t+C_{11},\\
&y=\frac{C_5}{4\omega}t\sin\omega t-\frac{C_6}{4\omega}t\cos\omega t+\\
&\qquad+C_7\cos\omega t+C_8\sin\omega t
\end{aligned}\right\} \tag{19}
$$

with

$$
\left\{\begin{aligned}
&K=\sqrt{\alpha_3^2-\beta_1},\quad \omega=\sqrt{\gamma_1},\\
&M=\frac{1}{2}\left(\frac{\alpha_3^2 C_2}{K}+\frac{\alpha_2\alpha_3 C_1}{K^2}+KC_2\right),\\
&N=-\frac{1}{2}\left(\frac{\alpha_3^2 C_1}{K}-\frac{\alpha_2\alpha_3 C_2}{K^2}+KC_1\right).
\end{aligned}\right. \tag{20}
$$

Expressions (19) contain 13 arbitrary constants $C_1, C_2, \ldots, C_{13}$. They are determined from the initial conditions

$$
\left\{\begin{aligned}
&x(0)=x_0, \quad V_x(0)=V_{x0},\\
&z(0)=z_0, \quad V_z(0)=V_{z0},\\
&y(0)=y_0, \quad V_y(0)=V_{y0},
\end{aligned}\right. \tag{21}
$$

the contact conditions

$$
\left\{\begin{aligned}
&x(\tau)=0, \quad V_x(\tau)=0,\\
&z(\tau)=0, \quad V_z(\tau)=0,\\
&y(\tau)=0, \quad V_y(\tau)=0
\end{aligned}\right. \tag{22}
$$

and the additional conditions

$$C_{11} = -\frac{\alpha_3 C_3}{2K^4}, \tag{23}$$

The latter condition is the result of the fact that these solutions obtained must satisfy the tenth equation (4), which has been differentiated [2]. The following expressions are found for the thirteen constants from equations (21), (22), and (23).

$$\left\{\begin{aligned}
&C_1 = \frac{\delta_1}{\delta_0}, \quad C_2 = \frac{\delta_2}{\delta_0}, \quad C_3 = \frac{\delta_3}{\delta_0},\\
&C_4 = -\frac{1}{B_7}(B_4 C_1 + B_5 C_2 + B_6 C_3 - B_8),\\
&C_5 = \frac{B_3}{A_{19}}, \quad C_6 = 4\omega\left(\omega\frac{B_2}{B_1} - V_{y0}\right),\\
&C_7 = y_0, \quad C_8 = \frac{B_2}{B_1}, \quad C_9 = V_{z0} + \frac{\alpha_3}{2K^2} C_4,\\
&C_{10} = K A_3 C_1 - \frac{\alpha_2 \alpha_3}{4K^4} C_2 - \frac{\alpha_3 C_3}{2K^3} - K z_0,\\
&C_{11} = -\frac{\alpha_3 C_3}{2K^4},\\
&C_{12} = V_{x0} - A_4 C_1 - A_{16} C_2 - \frac{\alpha_2}{K^2} C_9 + \frac{\alpha_3}{K} C_{10},\\
&C_{13} = x_0 - A_1 C_1 + A_2 C_2 + \frac{\alpha_3}{K^2} C_9 + \frac{\alpha_2}{K^3} C_{10}.
\end{aligned}\right. \tag{24}$$

The following relationships are used in these expressions:

$$\begin{aligned}
\delta_0 &= D_1 D_6 D_{11} + D_2 D_7 D_9 + D_3 D_5 D_{10} -\\
&\quad - D_1 D_7 D_{10} - D_2 D_5 D_{11} - D_3 D_6 D_9,\\
\delta_1 &= D_2 D_7 D_{12} + D_3 D_8 D_{10} + D_4 D_6 D_{11} -\\
&\quad - D_2 D_8 D_{11} - D_3 D_6 D_{12} - D_4 D_7 D_{10},\\
\delta_2 &= D_1 D_8 D_{11} + D_3 D_5 D_{12} + D_4 D_7 D_9 -\\
&\quad - D_1 D_7 D_{12} - D_3 D_8 D_9 - D_4 D_5 D_{11},\\
\delta_3 &= D_1 D_6 D_{12} + D_2 D_8 D_9 + D_4 D_5 D_{10} -\\
&\quad - D_1 D_8 D_{10} - D_2 D_5 D_{12} - D_4 D_6 D_9,
\end{aligned}$$

$$D_1 = B_7 B_9 - B_4 B_{12}, \qquad D_2 = B_7 B_{10} - B_5 B_{12},$$
$$D_3 = B_7 B_{11} - B_6 B_{12}, \qquad D_4 = B_7 B_{13} - B_8 B_{12},$$
$$D_5 = B_7 B_{14} - B_4 B_{17}, \qquad D_6 = B_7 B_{15} - B_5 B_{17},$$
$$D_7 = B_7 B_{16} - B_6 B_{17}, \qquad D_8 = B_7 B_{18} - B_8 B_{17},$$
$$D_9 = B_7 B_{19} - B_4 B_{22}, \qquad D_{10} = B_7 B_{20} - B_5 B_{22},$$
$$D_{11} = B_7 B_{21} - B_6 B_{22}, \qquad D_{12} = B_7 B_{23} - B_8 B_{22},$$

$$B_1 = A_{19} (\omega \tau \cos \omega \tau - \sin \omega \tau) + \\ + \left(\omega A_{20} + \frac{\cos \omega \tau}{4}\right) \tau \sin \omega \tau,$$

$$B_2 = A_{19} (\tau V_{y0} + y_0) \cos \omega \tau + \\ + \left(V_{y0} A_{20} + \frac{y_0 \sin \omega \tau}{4}\right) \tau \sin \omega \tau,$$

$$B_3 = 4 \omega V_{y0} A_{20} + \omega y_0 \sin \omega \tau - \\ - \omega \frac{B_2}{B_1} (\cos \omega \tau + 4 \omega A_{20}),$$

$$B_4 = - A_1 + A_3 \left(\frac{\alpha_2}{K^2} + \alpha_3 \tau - K A_{10}\right) - A_4 \tau + A_5,$$

$$B_5 = A_2 + A_6 - A_{16} \tau - \frac{\alpha_2 \alpha_3}{4 K^5} \left(\frac{\alpha_2}{K^2} + \alpha_3 \tau - K A_{10}\right),$$

$$B_6 = A_7 - \frac{\alpha_3}{2 K^4} \left(\frac{\alpha_2}{K^2} + \alpha_3 \tau - K A_{10}\right) + \frac{\alpha_2 \alpha_3 \tau^2}{4 K^4},$$

$$B_7 = A_8 - \frac{\alpha_3}{2 K^4} (\alpha_2 \tau - \alpha_3 - K^2 A_9),$$

$$B_8 = - x_0 + z_0 \left(\frac{\alpha_2}{K^2} + \alpha_3 \tau - K A_{10}\right) - \tau V_{z0} + \\ + \frac{V_{z0}}{K^2} (\alpha_2 \tau - \alpha_3 - K^2 A_9),$$

$$B_9 = A_{11} - A_3 \cos K \tau,$$

$$B_{10} = A_{12} + \frac{\alpha_2 \alpha_3}{4 K^5} \cos K \tau,$$

$$B_{11} = \frac{\alpha_3 \tau^2}{4 K^2} - \frac{\alpha_3}{2 K^4} (1 - \cos K \tau),$$

$$B_{12} = \frac{\alpha_3}{2 K^3} (\sin K \tau - K \tau),$$

$$B_{13} = - z_0 \cos K \tau - \frac{V_{z0}}{K} \sin K \tau,$$

$$B_{14} = A_3 (\alpha_3 + K^2 A_9) - A_4 + A_{13},$$

$$B_{15} = -\frac{\alpha_2 \alpha_3}{4 K^5} (\alpha_3 + K^2 A_9) + A_{14} - A_{16},$$

$$B_{16} = -A_8 - \frac{\alpha_3}{2 K^4} (\alpha_3 + K^2 A_9) + \frac{\alpha_2 \alpha_3 \tau}{2 K^4},$$

$$B_{17} = A_{15} - \frac{\alpha_3}{2 K^2} \left(\frac{\alpha_2}{K^2} - K A_{10} \right),$$

$$B_{18} = z_0 (\alpha_3 + K^2 A_9) - V_{z0} + V_{z0} \left(\frac{\alpha_2}{K^2} - K A_{10} \right),$$

$$B_{19} = A_{17} + K A_3 \sin K \tau,$$

$$B_{20} = A_{18} - \frac{\alpha_2 \alpha_3}{4 K^4} \sin K \tau,$$

$$B_{21} = -\frac{\alpha_3}{2 K^3} (\sin K \tau - K \tau),$$

$$B_{22} = -\frac{\alpha_3}{2 K^2} (1 - \cos K \tau),$$

$$B_{23} = K z_0 \sin K \tau - V_{z0} \cos K \tau,$$

$$A_1 = \frac{\alpha_2 \alpha_3^2}{4 K^6} + \frac{5 \alpha_2}{4 K^4},$$

$$A_2 = \frac{3 \alpha_2^2 \alpha_3}{4 K^7} + \frac{\alpha_3^3}{2 K^5},$$

$$A_3 = \frac{\alpha_3^2}{4 K^4} + \frac{1}{4 K^2},$$

$$A_4 = \frac{\alpha_2^2 \alpha_3}{2 K^6} + \frac{\alpha_3^3}{4 K^4} - \frac{\alpha_3}{4 K^2},$$

$$\begin{aligned} A_5 = {} & \frac{\alpha_2}{4 K^3} \tau \sin K \tau - \\ & - \left(\frac{\alpha_2^2 \alpha_3}{4 K^6} + \frac{\alpha_3^3}{4 K^4} + \frac{\alpha_3}{4 K^2} \right) \tau \cos K \tau + \\ & + \left(\frac{3 \alpha_2^2 \alpha_3}{4 K^7} + \frac{\alpha_3^3}{2 K^5} \right) \sin K \tau + \\ & + \left(\frac{\alpha_2 \alpha_3^2}{4 K^6} + \frac{5 \alpha_2}{4 K^4} \right) \cos K \tau, \end{aligned}$$

$$\begin{aligned} A_6 = {} & - \left(\frac{\alpha_2^2 \alpha_3}{4 K^6} + \frac{\alpha_3^3}{4 K^4} + \frac{\alpha_3}{4 K^2} \right) \tau \sin K \tau - \\ & - \frac{\alpha_2}{4 K^3} \tau \cos K \tau + \left(\frac{\alpha_2 \alpha_3^2}{4 K^6} + \frac{5 \alpha_2}{4 K^4} \right) \sin K \tau - \\ & - \left(\frac{3 \alpha_2^2 \alpha_3}{4 K^7} + \frac{\alpha_3^3}{2 K^5} \right) \cos K \tau, \end{aligned}$$

$$A_7 = -\frac{\alpha_2 \alpha_3}{48 K^2} \tau^4 + \left(\frac{\alpha_3^2}{12 K^2} - \frac{1}{12} \right) \tau^3,$$

$$A_8 = \frac{\alpha_2 \alpha_3}{12 K^2} \tau^3 - \left(\frac{\alpha_3^2}{4 K^2} - \frac{1}{4} \right) \tau^2,$$

$$A_9 = \frac{\alpha_2}{K^3} \sin K\tau - \frac{\alpha_3}{K^2} \cos K\tau\,,$$

$$A_{10} = \frac{\alpha_3}{K^2} \sin K\tau - \frac{\alpha_2}{K^3} \cos K\tau\,,$$

$$A_{11} = \left(\frac{\alpha_3^2}{4K^3} + \frac{1}{4K}\right)\tau \sin K\tau - \frac{\alpha_2\alpha_3}{4K^4}\tau \cos K\tau + \\ + \frac{\alpha_2\alpha_3}{4K^5} \sin K\tau + \left(\frac{\alpha_3^2}{4K^4} + \frac{1}{4K^2}\right)\cos K\tau\,,$$

$$A_{12} = -\frac{\alpha_2\alpha_3}{4K^4}\tau \sin K\tau - \left(\frac{\alpha_3^2}{4K^3} + \frac{1}{4K}\right)\tau \cos K\tau + \\ + \left(\frac{\alpha_3^2}{4K^4} + \frac{1}{4K^2}\right)\sin K\tau - \frac{\alpha_2\alpha_3}{4K^5}\cos K\tau\,,$$

$$A_{13} = \left(\frac{\alpha_2^2\alpha_3}{4K^5} + \frac{\alpha_3^3}{4K^3} + \frac{\alpha_3}{4K}\right)\tau \sin K\tau + \\ + \frac{\alpha_2}{4K^2}\tau \cos K\tau - \\ - \left(\frac{\alpha_2\alpha_3^2}{4K^5} + \frac{\alpha_2}{K^3}\right)\sin K\tau + \\ + \left(\frac{\alpha_2^2\alpha_3}{2K^6} + \frac{\alpha_3^3}{4K^4} - \frac{\alpha_3}{4K^2}\right)\cos K\tau\,,$$

$$A_{14} = \frac{\alpha_2}{4K^2}\tau \sin K\tau - \\ - \left(\frac{\alpha_2^2\alpha_3}{4K^5} + \frac{\alpha_3^3}{4K^3} + \frac{\alpha_3}{4K}\right)\tau \cos K\tau + \\ + \left(\frac{\alpha_2^2\alpha_3}{2K^6} + \frac{\alpha_3^3}{4K^4} - \frac{\alpha_3}{4K^2}\right)\sin K\tau + \\ + \left(\frac{\alpha_2\alpha_3^2}{4K^5} + \frac{\alpha_2}{K^3}\right)\cos K\tau\,,$$

$$A_{15} = \frac{\alpha_2\alpha_3}{4K^2}\tau^2 - \left(\frac{\alpha_3^2}{2K^2} - \frac{1}{2}\right)\tau\,,$$

$$A_{16} = \frac{\alpha_2\alpha_3^2}{4K^5} + \frac{\alpha_2}{K^3}\,,$$

$$A_{17} = \frac{\alpha_2\alpha_3}{4K^3}\tau \sin K\tau + \left(\frac{\alpha_3^2}{4K^2} + \frac{1}{4}\right)\tau \cos K\tau\,,$$

$$A_{18} = \left(\frac{\alpha_3^2}{4K^2} + \frac{1}{4}\right)\tau \sin K\tau - \frac{\alpha_2\alpha_3}{4K^3}\tau \cos K\tau\,,$$

$$A_{19} = \frac{\tau \cos \omega\tau}{4} + \frac{\sin \omega\tau}{4\omega}\,,$$

$$A_{20} = \frac{\tau \sin \omega\tau}{4} - \frac{\cos \omega\tau}{4\omega}\,.$$

4. Investigation of the Optimum Rendezvous Maneuver based on the Non-linear Equations of Motion of the Powered Spacecraft (non-linear theory)

If the non-linear equations of motion of the powered spacecraft are used (non-linear theory), then the variational problem of the optimum rendezvous maneuver for minimum fuel consumption amounts to determining the minimum of the following integrals, as was shown in [2]

$$J = \int_0^\tau (a_x^2 + a_z^2 + a_y^2)\, dt \qquad (1')$$

with the conditions

$$\left.\begin{aligned}
\Phi_1^* &= \frac{dx}{dt} - V_x = 0, \\
\Phi_2^* &= \frac{dV_x}{dt} - \frac{\mu p}{r_{0m}^4} x - 2 V_z \frac{\sqrt{\mu p}}{r_{0m}^2} + \\
&\quad + \frac{2\mu\varepsilon}{r_{0m}^3} z \sin\varphi_m + \\
&\quad + \mu \frac{x}{[x^2 + (r_{0m}+z)^2 + y^2]^{3/2}} - a_x = 0, \\
\Phi_3^* &= \frac{dz}{dt} - V_z = 0, \\
\Phi_4^* &= \frac{dV_z}{dt} - \frac{\mu}{r_{0m}^2} - \frac{\mu p}{r_{0m}^4} z + \\
&\quad + 2 V_x \frac{\sqrt{\mu p}}{r_{0m}^2} - \frac{2\mu\varepsilon}{r_{0m}^3} x \sin\varphi_m + \\
&\quad + \mu \frac{r_{0m} + z}{[x^2 + (r_{0m}+z)^2 + y^2]^{3/2}} - a_z = 0, \\
\Phi_5^* &= \frac{dy}{dt} - V_y = 0, \\
\Phi_6^* &= \frac{dV_y}{dt} + \\
&\quad + \mu \frac{y}{[x^2 + (r_{0m}+z)^2 + y^2]^{3/2}} - a_y = 0
\end{aligned}\right\} \qquad (25)$$

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As also stated in [2], in this formulation, the optimum /152
variational laws for the motion parameters of the powered spacecraft are derived using an iteration method, and the initial values of the Lagrange multipliers $\lambda_{10}, \lambda_{20}, \ldots, \lambda_{60}$ are taken from the linear theory.

Just as for circular trajectories [3] only the optimum control program is of special interest for the non-linear theory, i. e. for the determination of the thrust acceleration components.

In order to compare the values derived from the linear and non-linear theory, we will now give the values of the first approximation of the thrust acceleration components derived in [2]:

$$\left\{\begin{aligned} a_x^{(1)} &= \frac{\lambda_2^{(1)}}{2} = \frac{1}{2}\left[\lambda_{20} + \left(-\lambda_{10} + 2\,\lambda_{40}\,\frac{\sqrt{\mu p}}{r_{0m}^2}\right)t\right], \\ a_z^{(1)} &= \frac{\lambda_4^{(1)}}{2} = \frac{1}{2}\left[\lambda_{40} - \left(2\,\lambda_{20}\,\frac{\sqrt{\mu p}}{r_{0m}^2} + \lambda_{30}\right)t\right], \\ a_y^{(1)} &= \frac{\lambda_6^{(1)}}{2} = \frac{1}{2}\left(\lambda_{60} - \lambda_{50}\,t\right). \end{aligned}\right. \tag{26}$$

5. Numerical Example

Using an electronic data-processing computer, we calculated the thrust acceleration components a_x, a_z, a_y for different flight trajectories having eccentricities ε = 0.1, 0.05, 0.01, 0.005 according to the linear theory. The numerical values shown in Table I were used for the four variations, of which the first represents contact with the spacecraft and the other three represent approach. These are shown in Figures 3 to 6.

TABLE I

variation	r_{0p} [10^6 m]	p [10^6 m]	ε	φ_1 [°]	φ_2 [°]	μ [10^{14} m³/sec²]	
1	6,560	7,216	0,1	105	110	3,986	
2	6,578	6,907	0,05	105	130	3,986	
3	6,583	6,649	0,01	105	160	3,986	
4	6,616	6,649	0,005	105	200	3,986	
variation	x_0 [10^3 m]	z_0 [10^3 m]	y_0 [10^3 m]	V_{x0} [m/sec]	V_{z0} [m/sec]	V_{y0} [m/sec]	τ [sec]
1	3,6	1,2	1,0	− 80	− 25	− 20	86
2	10	4	3	− 140	− 40	− 30	397
3	25	15	10	− 180	− 70	− 40	824
4	80	50	25	− 220	− 130	− 60	821

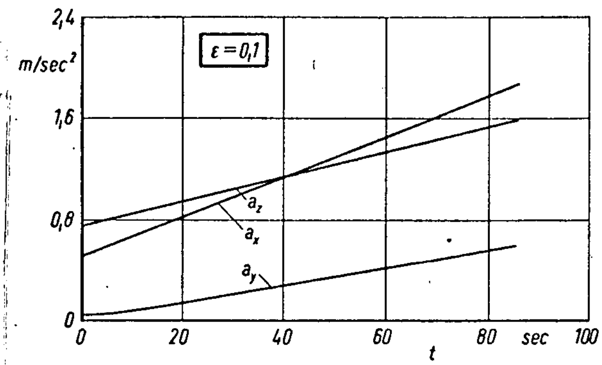

Fig. 3

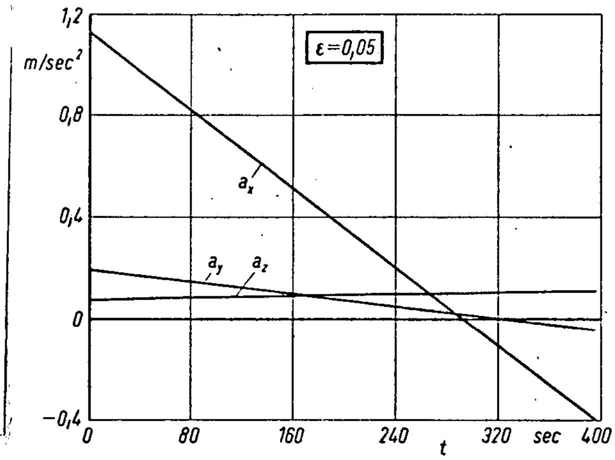

Fig. 4

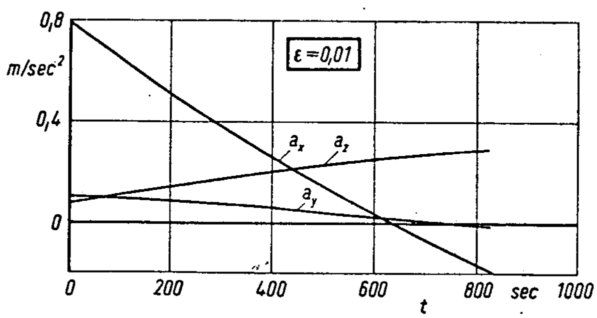

Fig. 5

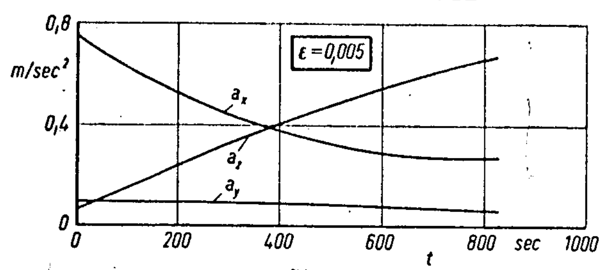

Fig. 6

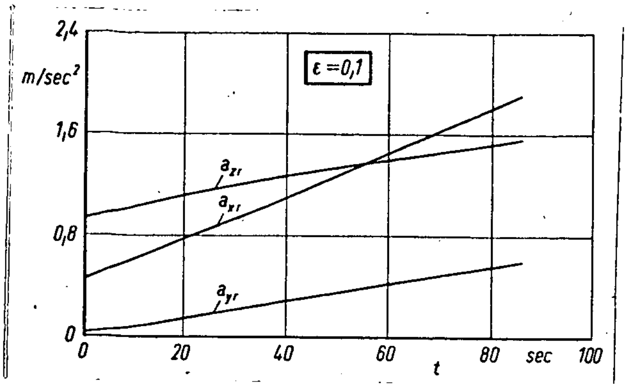

Fig. 7

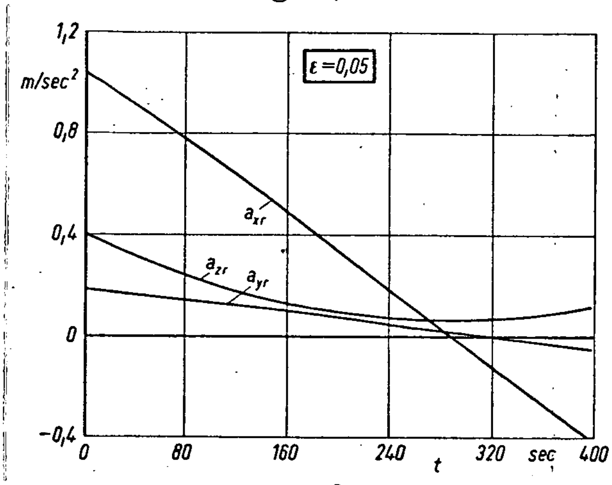

Fig. 8

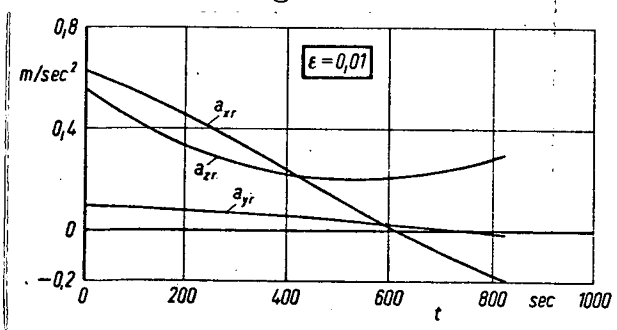

Fig. 9

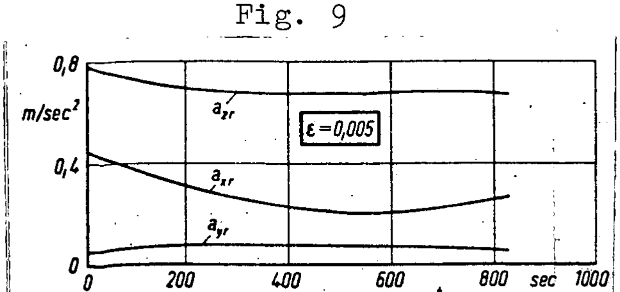

Fig. 10

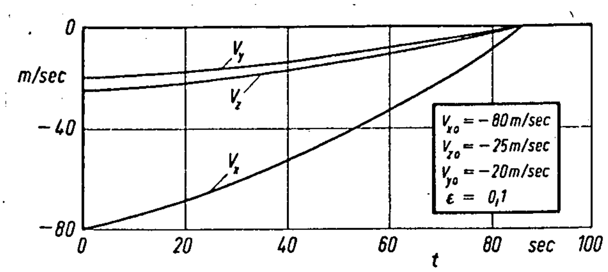

Fig. 11

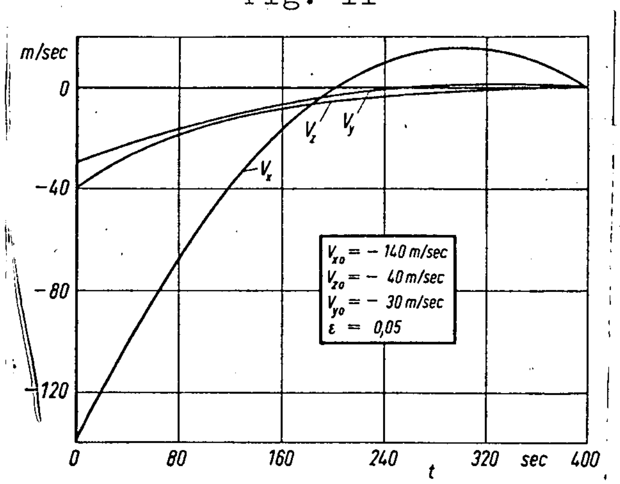

Fig. 12

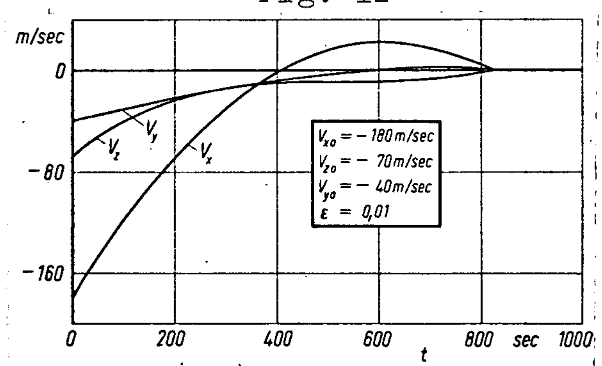

Fig. 13

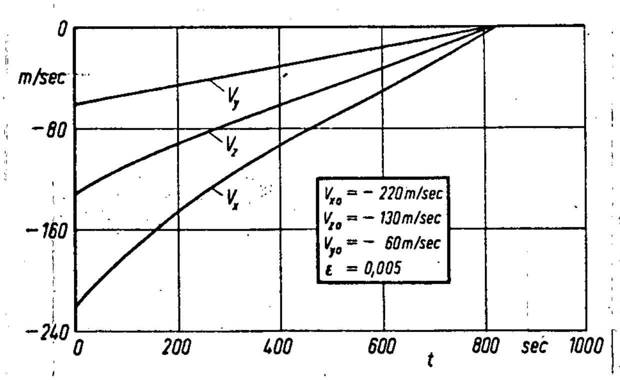

Fig. 14

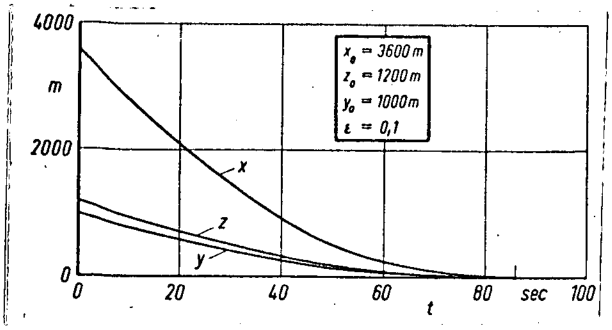

Fig. 15

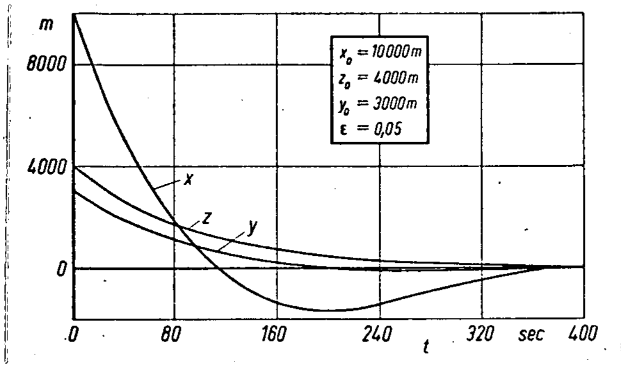

Fig. 16

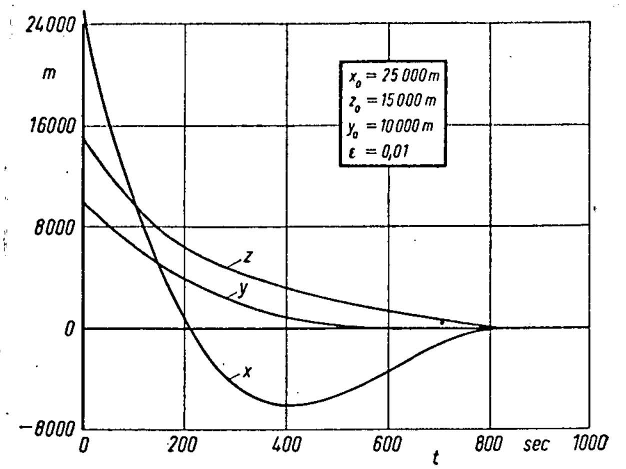

Fig. 17

Using these values, we determined the components of relative /153
acceleration of the powered spacecraft based on the equations of motion. This was plotted in in Figures 7 to 10.

The thrust acceleration components $a_x^{(1)}, a_z^{(1)}, a_y^{(1)}$ determined according to the non-linear theory as a first approximation are found to be only slightly different from those obtained according to the linear theory. This is why they are not shown here.

Using the linear theory we also determine the components /154
V_x, V_z, V_y of relative velocity of the powered spacecraft as well as its coordinates x, z, y. These are shown in Figures 11 through 14 and 15 through 18.

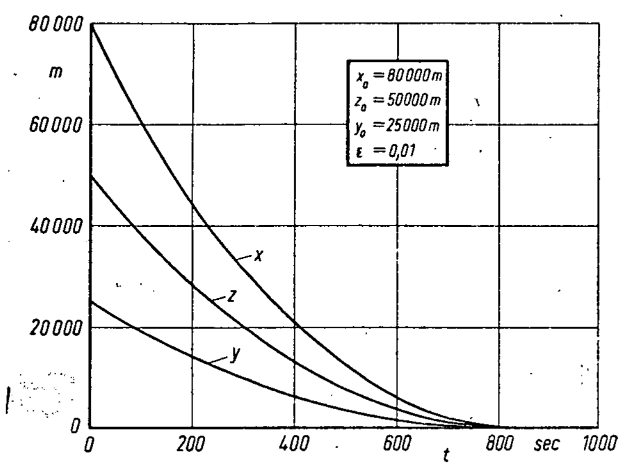

Fig. 18

Correction: Instead of ε=0.01 we should have written ε=0.005

6. FINAL REMARKS

The numerical calculations carried out for the optimum rendezvous maneuver for four different elliptical flight trajectories show that the thrust acceleration components a_x, a_z, a_y are linear for small burning times. For longer burning times, they deviate from this behaviour (Figures 3 to 6). For short burning times, the eccentricity of the flight trajectory influences the linear course of the thrust acceleration components in an insignificant way. However the influence is strong for long burning times.

In addition, for short burning times, the thrust acceleration components a_x, a_z, a_y dominate and are close to the values of the (total) relative acceleration components a_{xr}, a_{zr}, a_{yr}. Large differences between these values appear for long burning times (Figures 7 through 10). Also, we should note that the eccentricity of flight trajectories influences the linear course of the relative acceleration components a_{xr}, a_{zr}, a_{yr} more than those of the thrust acceleration a_x, a_z, a_y.

The components $V_x, V_z, V_y|$ of the relative velocity of the powered spacecraft and its coordinates x, y, z have a monotonic course, except for cases in which the target body is passed (Figures 11 through 14 and 15 through 18).

7. REFERENCES

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